

16 Functions 3

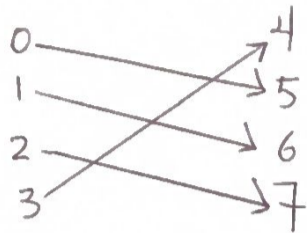
each $b \in B$ comes from exactly one input

each $b \in B$ comes from at most one output

each $b \in B$ comes from at least one input

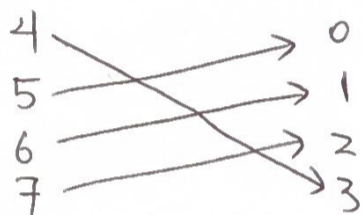
Def'n Function $f: A \rightarrow B$ is a bijection if f is one-to-one & onto.

Ex 1 $f: \{0, 1, 2, 3\} \rightarrow \{4, 5, 6, 7\}$ is a bijection: each domain element is paired with a codomain element and vice versa.



reverse all arrows to get inverse of f .

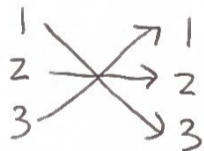
$$f^{-1}: \{4, 5, 6, 7\} \rightarrow \{0, 1, 2, 3\}$$



Def'n The inverse of a bijection $f: A \rightarrow B$ is the function $f^{-1}: B \rightarrow A$ defined by $f^{-1}(b) = a$ whenever $f(a) = b$.

Ex 2 Is $g: \{0, 1, 2, 3\} \rightarrow \{0, 1, 2, 3\}$ a bijection? If so, find its inverse.

A $0 \xrightarrow{g} 0$ is a bijection. Its inverse is itself.



Ex 3 Is $f: \mathbb{N} \rightarrow \mathbb{N}$, $f(x) = x^2 - x$ a bijection?

A $f(0) = 0^2 - 0 = 0$, $f(1) = 1^2 - 1 = 0$, but $0 \neq 1$, so f is not one-to-one.

Hence f is not a bijection.

Ex 4 Find inverse of $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 2x + 1$.

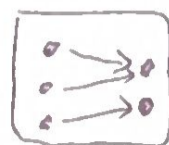
A Swap x and y in $y = 2x + 1$ and solve for y :

$$x = 2y + 1 \Rightarrow y = \frac{1}{2}(x - 1) \Rightarrow f^{-1}(x) = \frac{1}{2}(x - 1).$$

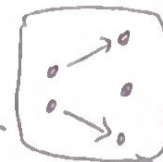
Pictures

In general ...

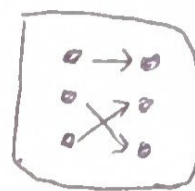
(a) $f: A \rightarrow B$ is onto $\Rightarrow |A| \geq |B|$



(b) $f: A \rightarrow B$ is one-to-one $\Rightarrow |A| \leq |B|$



(c) $f: A \rightarrow B$ is bijection $\Rightarrow |A| = |B|$



Ex Is $f: \{0, 1, 2, 3\} \rightarrow \{0, 1, 2\}$, $f(x) = 2x \pmod{3}$ one-to-one?

A No since codomain has size 3 while domain has size 4.

If we take the contrapositive of (b), we get:

Pigeonhole principle If $|B| > |A|$, then $f: A \rightarrow B$ is not one-to-one, so $f(a_1) = f(a_2)$ for some $a_1 \neq a_2$.

Ex The function $\{13 \text{ people}\} \rightarrow \{12 \text{ months of the year}\}$
 person \rightarrow their birth month

Cannot be one-to-one by the Pigeonhole Principle, so some 2 out of 13 people must share a birth month.